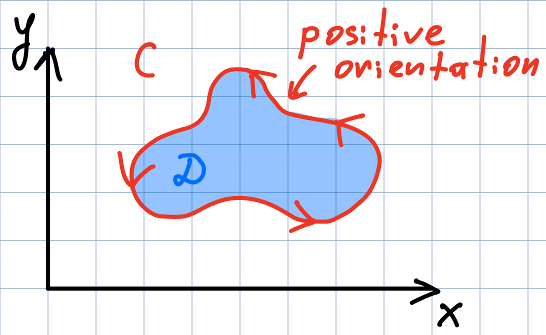
# LAST TIME:

## Green's theorem




Simple, closed curve
(not crossing itself)
with positive orientation (counterclockwise)
(region is always on the left as we go around C)
bounding the region $\mathcal{D}$.

**Green's THM:**

$$\int_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

positively oriented

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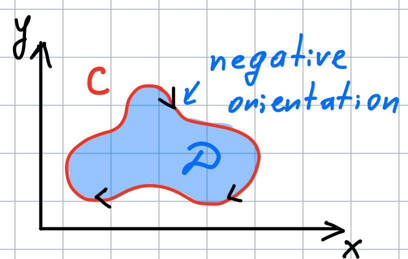
**Rmk:** For a negatively oriented curve $C$:




$$\int_C P\,dx + Q\,dy = -\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Write $\int_C x \sin z \, ds$ as an integral w.r.t. $t$

$$\int_C x \sin z \, ds = \int \qquad \, dt \quad \text{, where } C \text{ is given by...}$$

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Recall:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

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(a) $\begin{cases} x = \cos t \\ y = \sin t \\ z = 2t \end{cases}, \quad 0 \le t \le \pi$

$$\int_0^{\pi} \cos t \sin 2t \underbrace{\sqrt{(-\sin t)^2 + \cos^2 t + 2^2}}_{=\sqrt{5}}\,dt$$

(b) $\begin{cases} x = 2t+1 \\ y = t-1 \\ z = 3t \end{cases}, \quad 0 \le t \le 1$

$$\int_0^{1} (2t+1)\sin 3t \underbrace{\sqrt{2^2+1^2+3^2}}_{=\sqrt{14}}\,dt$$

(c) $\begin{cases} x = t \\ y = t^2 \\ z = t + \frac{\pi}{2} \end{cases}, \quad 1 \le t \le \pi$

$$\int_1^{\pi} t \sin\left(t+\tfrac{\pi}{2}\right) \underbrace{\sqrt{1^2+(2t)^2+1^2}}_{=\sqrt{4t^2+2}}\,dt$$

(d) $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}, \quad 0 \le t \le 2\pi$

$$\int_0^{2\pi} \cos t \sin t \underbrace{\sqrt{(-\sin t)^2+\cos^2 t+1}}_{=\sqrt{2}}\,dt$$

# Curl and divergence

$\vec{\nabla} = \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ - vector differential operator

$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$ - gradient

scalar function

**Curl:** For $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ - vector field,

$$\text{curl}\,\vec{F} = \text{"}\nabla \times \vec{F}\text{"} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

mnemonic

**Ex:** $\vec{F} = \langle xz, xyz, -y^2 \rangle$ Find curl $\vec{F}$

**Sol:**

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \langle -2y - xy, x - 0, yz - 0 \rangle = \langle -2y - xy, x, yz \rangle$$

⚠ $\operatorname{curl}(\nabla f) = \vec{0}$ for any function $f$.

$$\left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{matrix} \right| = \langle (f_z)_y - (f_y)_z \,,\, (f_x)_z - (f_z)_x \,,\, (f_y)_x - (f_x)_y \rangle = \langle 0, 0, 0 \rangle$$

Thus, if $\vec{F}$ is conservative, then $\operatorname{curl} \vec{F} = 0$.

Moreover, for $\vec{F}$ on entire $\mathbb{R}^3$,

if $\operatorname{curl} \vec{F} = 0$ then $\vec{F}$ is conservative.

Ex: $\vec{F}$ from example 1 has non-zero curl

$\Rightarrow \vec{F}$ is NOT conservative

**Ex:** $\vec{F} = \langle \underbrace{y^2z^3}_{P}, \underbrace{2xyz^3}_{Q}, \underbrace{3xy^2z^2}_{R} \rangle$

a) Show that $\vec{F}$ is conservative

b) Find $f$ such that $\vec{F} = \nabla f$.

**Sol:** (a) $\operatorname{curl} \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$= \langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \rangle = \vec{0}$$

$\Rightarrow \vec{F}$ is conservative

(b) $\begin{cases} f_x = y^2z^3 \\ f_y = 2xyz^3 \\ f_z = 3xy^2z^2 \end{cases} \xRightarrow{\int \text{w.r.t. } x} f = xy^2z^3 + g(y,z) \Rightarrow f_y = 2xyz^3 + g_y(y,z)$

$\Rightarrow g_y = 0 \Rightarrow g = h(z)$

$f = xy^2z^3 + h(z) \Rightarrow f_z = 3xy^2z^2 \Rightarrow h_z = 0 \Rightarrow h(z) = \underset{\text{const}}{K}$

$$\Rightarrow f(x,y,z) = xy^2z^3 + K$$

**Rmk:** If $\operatorname{curl} \vec{F} = 0$ at $(x_0, y_0, z_0)$,
then $\vec{F}$ is called *irrotational* at $(x_0, y_0, z_0)$.

(More details of curl interpretation: Stoke's THM (later in the course))

**Divergence:**

$$\operatorname{div} \vec{F} = \text{"}\nabla \cdot \vec{F}\text{"} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad \text{- divergence of } \vec{F}(x,y,z).$$

$\vec{F} = \langle P, Q, R \rangle$ mnemonic

**Ex:** $\vec{F} = \langle xy, xyz, -y^2 \rangle$. Find $\operatorname{div} \vec{F}$.

**Sol:** $$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} xyz + \frac{\partial}{\partial z}(-y^2) = y + xz + 0$$

$\operatorname{div}(\operatorname{curl} \vec{F}) = 0$ for any vector-field $\vec{F}$.

// $\operatorname{curl} \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$\operatorname{div}(\operatorname{curl} \vec{F}) = (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}) = 0. \quad //$$

**Ex:** $\vec{F}$ from the previous example CANNOT be written as $\operatorname{curl} \vec{G}$ for some $\vec{G}$, since $\operatorname{div} \vec{F} \neq 0$.

Rmk: If $\text{div}\,\vec{F} = 0$, $\vec{F}$ is called incompressible.

(The reason of this interpr. follows from Divergence THM ← later in the course)

$\boxed{\text{div}\,\vec{F} = \text{rate of change of the mass of gas in a unit volume per unit of time.}}$

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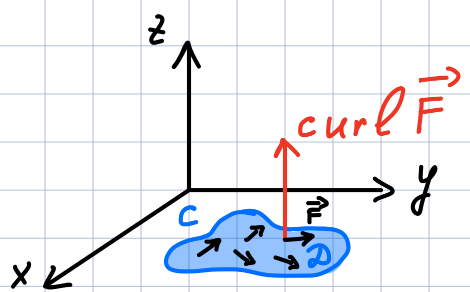
Let $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$ - indep. of $z$ and no $z$-component.

(vector field on $xy$-plane, extended to 3D)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \Rightarrow \vec{k} \cdot \text{curl}\,\vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Green's theorem: $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = \int_D \underbrace{(\text{curl}\,\vec{F}) \cdot \vec{k}}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}\, dA$

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Rmk: $\text{div}\,\nabla f = "\nabla \cdot \nabla f" = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ – Laplace operator.